Exponentials and trigonometric functions

Saturday, September 9, 2023 9:04 AM

Definition.

$$\exp(z) = e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots$$

For *any* $R>0$, $\frac{R^n}{n!} \to 0$ as $n \to \infty$ (Prove it!). Thus $\exp(z)$ exists for any $z \in \mathbb{C}$, an entire function.

$$(e^z)' = \sum_{n=0}^{\infty} \left(\frac{z^n}{n!}\right)' = \sum_{n=1}^{\infty} \frac{n z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = e^z$$

Consider $f(z) = e^z e^{a-z}$ for a *fixed* $a \in \mathbb{C}$. Then:

$$f'(z) = e^z \cdot e^{a-z} + - e^z \cdot e^{a-z} = 0.$$

so $f(z) \equiv f(0)$ – a constant, $f(0) = e^a$.

So, we have, for *any* $a, z \in \mathbb{C}$: $e^z e^{a-z} = e^a$.

Let $w = a - z$. Then $\forall z, w \in \mathbb{C}$:

$$\boxed{e^z \cdot e^w = e^{z+w}}$$

If $z = x+iy$, then $e^z = e^x e^{iy}$.

Observe: $e^{\bar{z}} = \sum \frac{\bar{z}^n}{n!} = \overline{e^z}$. So $|e^z|^2 = e^z \cdot e^{\bar{z}} = e^{2x}$,

so $|e^z| = e^x$, $|e^{iy}| = 1$.

e^{iy} lies on the unit circle.

$$e^{iy} = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n+1}}{(2n+1)!} = \cos y + i \sin y = \operatorname{cis}(y)$$

Euler's formula.

Leonard Euler

$$\boxed{y \in \arg e^z}$$

In particular, $\boxed{e^{\pi i} = -1}$.

$e^{2\pi i} = 1$, so $e^z = e^{z + 2k\pi i}$ – exponent is $2\pi i$-periodic.

Logarithm.

e^w – the complex number with $|e^w| = e^{\operatorname{Re} w}$, $\operatorname{Im} w \in \arg w$.

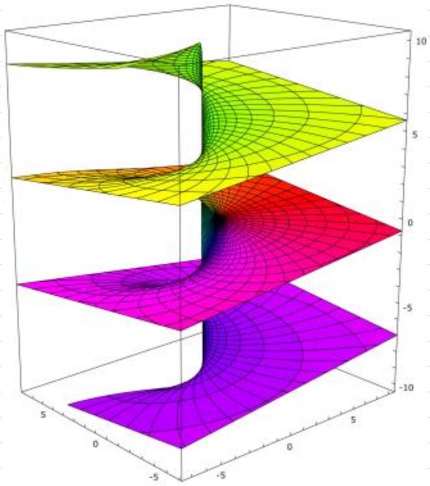
Solve $z = e^w$

$z = 0$ – no solutions! ($\log 0$ – not defined).

$z \neq 0$. $\operatorname{Re} w = \log|z|$ – the usual logarithm of a positive $|z|$

$\operatorname{Im} w \in \arg z$ – many possible values!

$\log z = \log|z| + i \arg z$ - multivalued function!



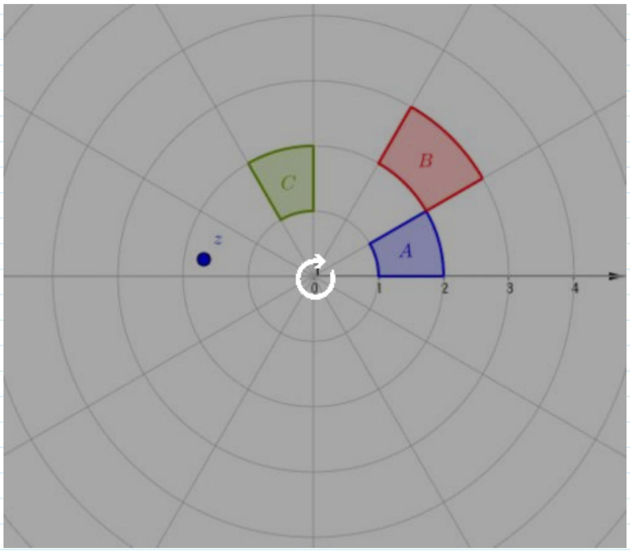
$\operatorname{Log} z := \log|z| + i \operatorname{Arg} z$ - principle value of logarithm. $\pi \geq \operatorname{Arg} z > -\pi$
Not continuous on $\{\operatorname{Re} z < 0, \operatorname{Im} z = 0\}$.

Remark. $e^{\log z} = z$ but $\log e^z = \{z + 2\pi i \cdot k,\ k \in \mathbb{Z}\}$.

Def. A continuous function $\ell(z)$ on a set K is called a branch of $\log z$ if $\forall z: \ell(z) \in \log z$.

Example $\ell(z) = \operatorname{Log} z$ is a branch on the set $\{z : -\pi < \operatorname{Arg} z < \pi\} = \mathbb{C} \setminus \mathbb{R}_-$.

[Complex logarithm map (principal branch)]



<u>Theorem.</u> If $\ell(z)$, a branch of log, is defined on $B(z_0, r)$, then $\ell(z)$ is analytic on $B(z_0, r)$, and $\ell'(z) = \frac{1}{z}$.

<u>Proof.</u> Analyticity – from homework problem. Also

$$e^{\ell(z)} = z \Rightarrow \ell'(z) = \frac{1}{(e^w)'|_{w=\ell(z)}} = \frac{1}{z} \blacksquare$$

<u>Theorem.</u> In $\mathbb{D}\ (= B(0,1))$,

$$\mathrm{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n}.$$

<u>Proof.</u> Let $\ell(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n}$. Then $\ell'(z) = \sum_{n=0}^{\infty} (-1)^n z^n = \frac{1}{1+z} = (\mathrm{Log}(1+z))'$.

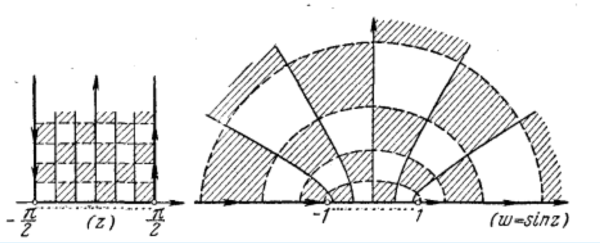
So $\ell(z) - \mathrm{Log}(1+z) \equiv \mathrm{const} = \ell(0) - \mathrm{Log}\, 1 = 0 \blacksquare$

Trigonometric and hyperbolic functions.

Return to Euler equations:

$$\left.\begin{array}{l} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{array}\right) \Rightarrow \begin{array}{l} \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} \end{array}$$

<u>Def.</u> $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ $\quad \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$.



Same power series: $\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$

$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$ 2π-periodic

$$\cos^2 z + \sin^2 z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2 = 1.$$

$$\cos' z = \left(\frac{e^{iz} + e^{-iz}}{2}\right)' = i\,\frac{e^{iz} - e^{-iz}}{2} = -\sin z.$$

$\sin' z = \cos z$

$$\tan z := \frac{\sin z}{\cos z} = -i\,\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$$ π-periodic

$$\tan(z+\pi) = -i\,\frac{e^{iz+i\pi} - e^{-iz-i\pi}}{e^{iz+i\pi} + e^{-iz-i\pi}} = -i\frac{(-1)}{(-1)}\tan z.$$

$$\tan' z = -i\,\frac{i(e^{iz} + e^{-iz})^2 - i(e^{iz} - e^{-iz})^2}{(e^{iz} + e^{-iz})^2} = \frac{4}{(e^{iz} + e^{-iz})^2} = \frac{1}{\cos^2 z}$$

Hyperbolic Functions:

$\cosh z := \frac{e^z + e^{-z}}{2} = \cos(iz)$

$\sinh z := \frac{e^z - e^{-z}}{2} = i \sin iz.$

$\operatorname{ctgh} z := \frac{\cosh z}{\sinh z}$

$\cosh^2 z - \sinh^2 z = 1$

$\cosh' = \sinh$

$\sinh' = \cosh$

$(\cosh t, \sinh t)$ - parameterization of hyperbola $x^2 - y^2 = 1$.

Inverse functions:

arctan z: multivalued!

$z = \tan w$

$z = -i\,\frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} \Rightarrow$

$$e^{2iw} = \frac{1+iz}{1-iz} \overset{\times i}{=} \frac{i-z}{i+z} \Rightarrow \boxed{w = -\frac{i}{2}\log\left(\frac{i-z}{i+z}\right)}$$

$$e^{2iw} = \frac{1+iz}{1-iz} \overset{\times i}{=} \frac{i-z}{i+z} \Rightarrow \boxed{w = -\frac{i}{2}\log\left(\frac{i-z}{i+z}\right)}$$

$z \neq \pm i$

Same way: $\operatorname{arcsinh} z = \log\left(z + \sqrt{z^2+1}\right)$.

$$z = \infty \Rightarrow -\frac{i}{2}\log(-1) = -\frac{i}{2}(\pi i + 2\pi k i) = \boxed{\frac{\pi}{2} + \pi k}$$